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HEREDITARY QI-RINGS(1)

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ABSTRACT. We consider in this paper rings in which every quasi-injective right R-module is injective. These rings will be referred to as right QI-rings. For a hereditary ring, this is equivalent to the condition that R be noetherian and a right V-ring. We also consider rings in which proper cyclic right R-modules are injective. These are right QI-rings which are either semisimple or right hereditary, right Ore domains in which indecomposable injective right R-modules are either simple or isomorphic to the injective hull of R_R.

Introduction. It has been shown that the following three conditions on a ring R are equivalent: (1) each simple right R-module is injective; (2) each right ideal is the intersection of maximal right ideals; (3) Rad $M = 0$ for all $M \in$ Mod-R. Rings satisfying any one of the above conditions are called *right V-rings* after Villamayor, who proved the equivalence of (1) and (2). In finding a counterexample to the conjecture that every V-ring is regular, Cozzens [3] produced an example of a noetherian V-ring in which every cyclic right R-module is either semisimple or free. This condition on the cyclics forces every quasi-injective right R-module to be injective. We shall call a ring in which every quasi-injective right R-module is injective a right *QI-ring*. In this paper we will consider the problem of characterizing hereditary QI-rings.

The class of finitely generated torsion right R-modules plays an important role in this consideration. We find that over a hereditary, noetherian, right V-ring, the class of finitely generated torsion right R-modules, the class of finitely generated semisimple right R-modules, and the class of finitely generated injective right R-modules are all synonymous. We use this equivalence to show that over a hereditary, noetherian ring R, R is a right V-ring if and only if R is a right QI-ring.

Following the example of Cozzens more closely, we also consider rings in which proper cyclic right R-modules are injective. We call these *right PCI-rings*. Every right PCI-ring is a right QI-ring. Further we show that a right PCI-ring is either semisimple or a right hereditary, right Ore domain in which every indecomposable injective right R-module is either simple or isomorphic to the injective hull of R_R.

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1. **Preliminaries.** Throughout this paper a ring R is associated with an identity element. Each R-module is unitary. Unless right or left is specified, a condition will be assumed to hold on both sides of the ring. If $M \in$ mod-R, then $E(M)$ denotes the injective hull of M_R.

An R-module M_R is *quasi-injective* if each homomorphism of any submodule N into M can be extended to a homomorphism of M into M. Semisimple right modules are clearly examples of quasi-injective modules. Johnson and Wong [8] have related injectives and quasi-injectives by proving that a module is quasi-injective if and only if it is closed under endomorphisms of its injective hull.

A submodule J_R is uniform if for any two submodules A_R and B_R, $A \cap B \neq 0$.

An element r of the ring R is called *regular* if $sr \neq 0$ and $rs \neq 0$ for any nonzero element s of R. For any right R-module M, an element $m \in M$ is a *torsion element* if $md = 0$ for some regular element $d \in R$. If every nonzero element of M is a torsion element, then M is said to be a *torsion module*; if no element of M is torsion, M is said to be *torsion-free*. Given a right R-module M, let tM denote the collection of torsion elements of M. In the case where R is a commutative ring, tM always forms a submodule of M. In the noncommutative case however, we can conclude that tM forms a submodule of M for any R-module M if and only if R has a classical right quotient ring (see Levy [10] and Gentile [6]).

A ring R is called a *right QI-ring* if every quasi-injective right R-module is injective.

A right R-module C is a *proper cyclic* in case $R \to C \to 0$ is exact but $0 \to R \to C \to 0$ is not. A ring R is called a *right PCI-ring* if every proper cyclic right R-module is injective.

Before proceeding to the main body of the work, we shall consider several lemmas which will prove useful later.

Lemma 1 (Kurshan [9]). *If R is a ring in which every semisimple right R-module is injective, then R is right noetherian.*

Lemma 2. *Let M be a quasi-injective right R-module. If M contains a copy of R, then M is injective.*

Proof. This is an obvious consequence of Baer's criterion for injectivity [1].

Lemma 3. *Let R be a noetherian, hereditary, semiprime ring and let $\mathfrak{F}_R$ denote the Serre class of finitely generated torsion right R-modules. Then* $\operatorname{Ext}^1_R(-, R)\colon \mathfrak{F}_R \to {}_R\mathfrak{F}$ *defines a duality.*

Proof. Let Q be the classical ring of right quotients of R. Then $\operatorname{Hom}_R(-, Q/R)$ and $\operatorname{Ext}^1_R(-, R)$ are naturally isomorphic on $\mathfrak{F}_R$. Using this it follows readily that the functor takes $\mathfrak{F}_R$ into ${}_R\mathfrak{F}$.

To complete the proof all we need show is that $\mathrm{Hom}_R(-, Q/R)$ is inverse to $\mathrm{Ext}^1_R(-, R)$ in $\mathfrak{F}_R$. Let $A \in \mathfrak{F}_R$. Then by Cartan and Eilenberg [2],

$$\mathrm{Hom}_R(\mathrm{Ext}^1_R(A, R), Q/R) \approx \mathrm{Tor}_1^R(A, \mathrm{Hom}_R(R, Q/R)) \approx \mathrm{Tor}_1^R(A, Q/R)$$

and by Sandomierski [12], $\mathrm{Tor}_1^R(A, Q/R) \approx A$, giving us the desired result.

2. **Hereditary right QI-rings.** In this section we will be concerned with characterizing hereditary right QI-rings.

Theorem 4. *Let R be a noetherian, hereditary right V-ring. Then finitely generated torsion right R-modules are semisimple and injective.*

Proof. Since Rad $R = 0$, R is semiprime and by Lemma 3 there exists a duality $\mathrm{Ext}^1_R(-, R)$: $\mathfrak{F}_R \to {}_R\mathfrak{F}$. Since $\mathfrak{F}_R$ is a noetherian category, ${}_R\mathfrak{F}$ is an artinian category, and hence a noetherian category, by this duality. Using the duality once more, since ${}_R\mathfrak{F}$ is a noetherian category, $\mathfrak{F}_R$ is an artinian category. Thus every finitely generated torsion right R-module contains a simple module which splits off since R is a right V-ring. Hence every finitely generated torsion module is semisimple.

Theorem 5. *Let R be a hereditary, left noetherian ring. Then R is a right QI-ring if and only if R is a right noetherian, right V-ring.*

Proof. Suppose R is a right noetherian, right V-ring. By Faith [4], R is a product of simple noetherian, right V-rings. Hence it suffices to consider a simple, noetherian hereditary right V-ring. Since R is noetherian, it contains a uniform finitely generated, right ideal U. Since R is hereditary and simple, U is a projective generator. By Mitchell [11] there exists a category equivalence Mod-R $\to$ Mod-End U_R where End U_R is an Ore domain by Goldie [7]. Since quasi-injectives and injectives are Morita invariants, it suffices to work in a hereditary, noetherian, right V-ring, right Ore domain.

Let M be a quasi-injective right R-module. Since R is noetherian, $E(M)$ is a direct sum of indecomposable, injective right submodules. By intersecting M with each of these indecomposable injectives, we get a direct sum of indecomposable quasi-injectives, the direct sum of which is isomorphic to M. Hence it suffices to show that indecomposable quasi-injective right R-modules are injective.

Let M be an indecomposable quasi-injective right R-module. Let $m \in M$. If $mR \approx R$ then M contains a copy of the ring and, by Lemma 2, M is injective. If $\mathrm{ann}_R(m) \neq 0$, then $mR \approx R/I$ for some right ideal I. Since R is a domain, I contains a regular element and hence R/I is a finitely generated torsion module. By Theorem 4, mR is injective and thus $mR = E(M)$ forcing M to be injective.

Conversely, suppose R is a right QI-ring. Since simple right R-modules are quasi-injective, R is a right V-ring. Since semisimples are also quasi-injective, R is right noetherian by Lemma 1.

We can take our knowledge of the structure of these hereditary, noetherian, right V-rings a step further by characterizing the class of finitely generated injectives.

Corollary 6. *Let R be a noetherian, hereditary, right V-ring. Then a finitely generated right R-module is injective if and only if it is semisimple.*

Proof. Since R is right noetherian and a right V-ring, every semisimple right R-module is injective.

Conversely, assume that M is a finitely generated injective right R-module. By Theorem 4, tM, the torsion submodule of M, is semisimple and injective. Thus $M \approx tM \oplus K$ where K is torsion-free. Suppose $K \neq 0$. As in the proof of Theorem 5, we have a category equivalence T: Mod-$R \to$ Mod-S where S is a hereditary, noetherian, right V-ring, right Ore domain. By Theorem 4, all finitely generated torsion modules are semisimple both in Mod-R and Mod-S. If $T(K)$ is not a torsion module, then $T(K)$ must contain a copy of S since this is the only cyclic in Mod-S which is not a torsion module. Since $T(K)$ is injective, it contains $E(S_S)$ which implies that $E(S_S)$ is finitely generated. Then by Faith and Walker [5], S is right artinian. However, every right V-ring is semiprime. So S must be semisimple by the Wedderburn-Artin theorem. R must also be semisimple in which case every right R-module is semisimple. Hence we may suppose that $K = 0$. Then $M = tM$ is semisimple.

3. **Right PCI-rings.** As was mentioned in the introduction, one of the motivating factors for this topic was the example of Cozzens which provided a nontrivial right QI-ring. Cozzens' example has the property that proper cyclic right R-modules are injective. We shall call rings with this property right PCI-rings.

Theorem 7. *If R is a right noetherian, right PCI-ring, then R is a right hereditary, right QI-ring.*

Proof. Let M be any right R-module with injective hull $E(M)$. Consider $E(M)/M$. If $\overline{m} \in E(M)/M$, then $\text{ann}_R(\overline{m}) \neq 0$ and is in fact a large right ideal. Thus $\overline{m}R$ is a proper cyclic and is injective by hypothesis. So $\overline{m}R$ splits off as a summand of $E(M)/M$. Take a maximal collection $\{\overline{m}_i R\}_{i \in I}$ of linearly independent cyclic summands of $E(M)/M$. Suppose $E(M) \neq \sum_{i \in I} \overline{m}_i R$. Since R is right noetherian, every direct sum of injectives is injective. Thus $\sum_{i \in I} \overline{m}_i R$ is injective and $E(M)/M \approx \sum_{i \in I} \overline{m}_i R \oplus A$ for some submodule A of $E(M)/M$. Every cyclic submodule C of A is proper and thus injective. So C splits off contradicting the maximality of $\{\overline{m}_i R\}_{i \in I}$. Hence $E(M)/M = \sum_{i \in I} \overline{m}_i R \approx \sum_{i \in I} \oplus \overline{m}_i R$ is a direct sum of proper cyclics and is therefore injective. Thus $0 \to M \to E(M) \to E(M)/M \to 0$ is an injective resolution for M. Thus inj dim $R_R \leq 1$ implying that R is right hereditary.

Since R is right noetherian, for any quasi-injective right R-module M we can get a decomposition of M into a direct sum of indecomposable quasi-injective right R-modules. Thus it suffices to show that indecomposable quasi-injectives are injective. Let M be an indecomposable quasi-injective right R-module and let $m \in M$. If $mR \approx R$, then M is injective by Lemma 2. Otherwise mR is a proper cyclic and is injective. Since M is indecomposable, $mR = M$ and M is injective. Thus R is a right QI-ring.

It can be observed from the above proof that every indecomposable injective right R-module is either cyclic or isomorphic to $E(R)$. Using the following lemma due to Faith, we can say even more about the structure of these rings.

Lemma 8. *Let R be a noetherian right PCI-ring. Then the endomorphism ring of an indecomposable injective right R-module is a field.*

Proof. Let M be an indecomposable injective right R-module. If M has a cyclic submodule C such that $C \approx R$, then $M \approx E(R)$. Thus $E(R)$ is indecomposable. R has a semisimple classical quotient ring Q and since the singular right ideal is zero, $Q \approx E(R)$. Thus $E(R)$ is a right and left semisimple artinian ring. Since it is indecomposable over R, $E(R)$ is a field. End $M \approx$ End $E(R) \approx E(R)$. Thus End M is a field.

Otherwise any nonzero submodule N of M contains a proper cyclic C. Since C is injective, $C = N = M$. Thus M has no proper submodules and is therefore simple. So End M is a field.

Corollary 9. *If R is a right noetherian, right PCI-ring, then either R is a semisimple ring or else R is a right hereditary, right Ore domain in which every indecomposable injective right R-module is either simple or isomorphic to the right quotient field $E(R)$.*

Proof. If R has no indecomposable injective right R-module isomorphic to $E(R)$, then as in the proof of Lemma 8, every indecomposable injective is simple, and hence given a right R-module M it is contained in a semisimple right R-module $E(M)$ and thus is semisimple.

If R is not semisimple, then there exists an indecomposable injective right R-module M such that $M \approx E(R)$. This implies that $E(R)$ is indecomposable. Since $E(R)$ is indecomposable as a ring, it is a field. Thus R is a right Ore domain and every indecomposable injective right R-module is either simple or isomorphic to $E(R)$.

Corollary 10. *Suppose R is not semisimple. Then the following conditions on a left noetherian, left hereditary ring R are equivalent:*

(1) *R is a right noetherian, right PCI-ring.*

(2) *R is a right hereditary, right QI-ring, right Ore domain.*

(3) *R is a right hereditary, right noetherian V-ring which is a right Ore domain.*

Proof. (1) *implies* (2). This follows from Theorem 7 and Corollary 10.

(2) *implies* (3). This follows from Theorem 5.

(3) *implies* (1). We want to show that proper cyclics are injective. By Theorem 4, finitely generated torsion right R-modules are injective. Let C be a proper cyclic, $C \approx R/I$ for some right ideal I. Since R is an Ore domain, $r \in I$ implies that r is regular and $cr = 0, c \in C$. Thus C is a finitely generated torsion module and is injective.

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